which was what we wanted. Equality holds if and only if n is square-free i.e. of the form $p_1p_2\cdots p_k$, where the p_i are distinct prime numbers.

S99. Let ABC be an acute triangle. Prove that

$$\frac{1-\cos A}{1+\cos A}+\frac{1-\cos B}{1+\cos B}+\frac{1-\cos C}{1+\cos C}\le\left(\frac{1}{\cos A}-1\right)\left(\frac{1}{\cos B}-1\right)\left(\frac{1}{\cos C}-1\right).$$

Proposed by Daniel Campos Salas, Costa Rica

Solution: Since

$$\frac{1-\cos A}{1+\cos A}=\tan^2\frac{A}{2},\frac{1}{\cos A}-1=\frac{2\tan^2\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

and

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

we have

$$\sum_{cyc}\frac{1-\cos A}{1+\cos A}\le\prod_{cyc}\left(\frac{1}{\cos A}-1\right)\iff\sum_{cyc}\tan^2\frac{A}{2}\le\prod_{cyc}\frac{2\tan^2\frac{A}{2}}{1-\tan^2\frac{A}{2}}.$$

Let $x=\tan\frac{A}{2},y=\tan\frac{B}{2},z=\tan\frac{C}{2}$. From the given, triangle ABC is acute hence $x,y,z\in(0,1)$ and the original inequality can be rewritten as

$$x^2+y^2+z^2\le\frac{8x^2y^2z^2}{(1-x^2)(1-y^2)(1-z^2)}$$
$$\iff\left(x^2+y^2+z^2\right)\left(1-x^2\right)\left(1-y^2\right)\left(1-z^2\right)\le 8x^2y^2z^2,\quad(1)$$

where $xy+yz+zx=1$. We will prove that inequality (1) holds for any nonnegative real numbers x,y,z such that $xy+yz+zx=1$. Assuming that $x+y+z=1$ in homogeneous form of inequality (1)

$$\left(x^2+y^2+z^2\right)\prod_{cyc}\left(xy+yz+zx-x^2\right)\le 8x^2y^2z^2\left(xy+yz+zx\right)\quad(2)$$

and letting $p = xy + yz + zx, q = xyz$ we obtain that $x^2 + y^2 + z^2 = 1 - 2p$, $x^2y^2 + y^2z^2 + z^2x^2 = p^2 - 2q$, $\prod_{cyc} \left(p - x^2\right) = 4p^3 - (p+q)^2$ and inequality (2) becomes

$$(1-2p)\left(4p^3 - (p+q)^2\right) \le 8pq^2$$
$$\iff 0 \le 8pq^2 + (1-2p)(p+q)^2 - 4p^3(1-2p). \quad (3)$$

Since $p = xy + yz + zx \le \dfrac{(x+y+z)^2}{3} = \dfrac{1}{3}$ and $8pq^2 + (1-2p)(p+q)^2 - 4p^3(1-2p)$ is increasing when $q \ge 0$, then it suffices to prove the inequality (3) for $0 \le p \le \dfrac{1}{3}$ and $q = q_*$, where q_* is lower bound for q, which is good enough to prove (3). Since the lower bound $\dfrac{4p-1}{9}$ for q, which gives us Schür's inequality

$$\sum_{cyc} x(x-y)(x-z) \ge 0$$
$$\iff 9xyz \ge 4(x+y+z)(xy+yz+zx) - (x+y+z)^3$$
$$\iff \frac{4p-1}{9} \le q,$$

isn't good enough, then we will find another, better lower bound for q. Let $L = \sum_{cyc} x^2y, R = \sum_{cyc} xy^2$ then

$$L + R = \sum_{cyc} xy(x+y) = p - 3q,$$

and

$$L \cdot R = \sum_{cyc} x^2y \cdot \sum_{cyc} xy^2 = \sum_{cyc} x^3y^3 + 3x^2y^2z^2 + xyz \sum_{cyc} x^3.$$

Since $\sum_{cyc} x^3 = 1 + 3q - 3p$ and $\sum_{cyc} x^3y^3 = p^3 + 3q^2 - 3pq$ then

$$L \cdot R = p^3 + 3q^2 - 3pq + 3q^2 + q(1 + 3q - 3p) = p^3 + 9q^2 - 6pq + q$$

and, therefore,

$$\begin{aligned}0 \le (L-R)^2 &= (p-3q)^2 - 4\left(p^3 + 9q^2 - 6pq + q\right) \\ &= p^2 - 6pq + 9q^2 - 4p^3 - 36q^2 + 24pq - 4q \\ &= p^2 - 4p^3 - 27q^2 - 4q + 18pq\end{aligned}$$

which is equivalent to

$$\left(q - \frac{9p-2}{27}\right)^2 - \frac{(1-3p)^3}{27} \le 0 \implies q_* \le q,$$

where $q_* = \dfrac{9p - 2 - 2(1-3p)\sqrt{1-3p}}{27}$. Let $t = \sqrt{1-3p}$ then $p = \dfrac{1-t^2}{3}$, $t \in [0,1] \iff p \in \left[0, \dfrac{1}{3}\right]$, $q_* = \dfrac{(1+t)^2(1-2t)}{27}$. Thus, we have

$$1 - 2p = \frac{1+2t^2}{3},\ p + q_* = \frac{2(1+t)\left(5 - 5t - t^2\right)}{27},$$

$$(1-2p)(p+q_*)^2 - 4p^3(1-2p) = \frac{4(1+t)^2}{27^2}\left(28t^4 - 44t^3 + 15t^2 + 4t - 2\right)$$

and, therefore,

$$\begin{aligned}&8pq^2 + (1-2p)(p+q_*)^2 - 4p^3(1-2p) \\ &\quad= t^2\left(7 - 32t + 42t^2 - 8t^3 - 8t^4\right) \\ &\quad= t^2(2t-1)^2\left(7 - 4t - 2t^2\right) \ge 0,\end{aligned}$$

because

$$7 - 4t - 2t^2 = 9 - 2(1+t)^2 \ge 9 - 2\left(1 + \frac{1}{2}\right)^2 = \frac{9}{2} > 0.$$

Remark: We would like to thank Arkady Alt for the solution to this problem which remained unsolved for a very long time.